

# Dynamical Twisting and the $b$ Ghost in the Pure Spinor Formalism

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After adding an RNS-like fermionic vector  $\psi^m$  to the pure spinor formalism, the non-minimal  $b$  ghost takes a simple form similar to the pure spinor BRST operator. The N=2 superconformal field theory generated by the  $b$  ghost and the BRST current can be interpreted as a “dynamical twisting” of the RNS formalism where the choice of which spin  $\frac{1}{2}$   $\psi^m$  variables are twisted into spin 0 and spin 1 variables is determined by the pure spinor variables that parameterize the coset  $SO(10)/U(5)$ .

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## 1. Introduction

The pure spinor formalism for the superstring [1] has the advantage over the Ramond-Neveu-Schwarz (RNS) formalism of being manifestly spacetime supersymmetric and has the advantage over the Green-Schwarz (GS) formalism of allowing covariant quantization. However, the worldsheet origin of the pure spinor formalism is mysterious since its BRST operator and  $b$  ghost do not arise in an obvious manner from gauge-fixing.

In the non-minimal pure spinor formalism, the BRST current and  $b$  ghost can be interpreted as twisted  $\hat{c} = 3$  N=2 superconformal generators [2]. But when expressed in terms of the d=10 superspace variables and the non-minimal pure spinor variables, the  $b$  ghost and the resulting N=2 superconformal transformations are extremely complicated. In fact, the nilpotency of the  $b$  ghost was only recently verified [3][4].

In this paper, it will be shown that the  $b$  ghost dramatically simplifies when expressed in terms of a fermionic vector  $\psi^m$  that is defined in terms of the other worldsheet variables. If one treats the ten  $\psi^m$  variables as independent variables, 5 of the 16  $\theta^\alpha$  variables of d=10 superspace (and their conjugate momenta) can be eliminated [5]. The remaining 11  $\theta^\alpha$  variables and their conjugate momenta transform as the worldsheet superpartners of the pure spinor variables. The resulting N=2 superconformal field theory generated by the  $b$  ghost and the BRST current can be interpreted as a “dynamically twisted” version of the RNS formalism.

In this dynamically twisted superconformal field theory, the N=2 generators are

$$T = -\frac{1}{2}\partial x^m \partial x_m - \frac{(\lambda \gamma_m \gamma_n \bar{\lambda})}{2(\lambda \bar{\lambda})} \psi^m \partial \psi^n + \dots, \quad (1.1)$$

$$b = \frac{(\lambda \gamma_m \gamma_n \bar{\lambda})}{2(\lambda \bar{\lambda})} \psi^m \partial x^n + \dots,$$

$$j_{BRST} = -\frac{(\lambda \gamma_m \gamma_n \bar{\lambda})}{2(\lambda \bar{\lambda})} \psi^n \partial x^m + \dots,$$

$$J = -\frac{(\lambda \gamma_m \gamma_n \bar{\lambda})}{2(\lambda \bar{\lambda})} \psi^m \psi^n + \dots,$$

where  $\lambda^\alpha$  and  $\bar{\lambda}_\alpha$  are the non-minimal pure spinor ghosts whose projective components parameterize the coset  $SO(10)/U(5)$  that describes different twistings. The remaining terms ... in (1.1) are determined by requiring that  $(\lambda^\alpha, \bar{\lambda}_\alpha)$  and their worldsheet superpartners transform in an N=2 supersymmetric manner.

So the resulting N=2 superconformal field theory is the sum of a dynamically twisted RNS superconformal field theory with an N=2 superconformal field theory for the pure spinor variables. This interpretation of the BRST operator and the  $b$  ghost as coming from dynamical twisting of an N=1 superconformal field theory will hopefully lead to a better geometrical understanding of the pure spinor formalism.

In section 2, the non-minimal pure spinor formalism is reviewed. In section 3, the  $b$  ghost in the pure spinor formalism is shown to simplify when expressed in terms of an RNS-like  $\psi^m$  variable. In section 4, dynamical twisting of the RNS formalism will be defined and the resulting twisted N=2 superconformal generators will be related to the  $b$  ghost and BRST current in the pure spinor formalism. And in section 5, the results will be summarized.

## 2. Review of Non-Minimal Pure Spinor Formalism

As discussed in [2], the left-moving contribution to the worldsheet action in the non-minimal pure spinor formalism is

$$S = \int d^2z \left[ -\frac{1}{2} \partial x^m \bar{\partial} x_m - p_\alpha \bar{\partial} \theta^\alpha + w_\alpha \bar{\partial} \lambda^\alpha + \bar{w}^\alpha \bar{\partial} \bar{\lambda}_\alpha - s^\alpha \bar{\partial} r_\alpha \right] \quad (2.1)$$

where  $x^m$  and  $\theta^\alpha$  are d=10 superspace variables for  $m = 0$  to 9 and  $\alpha = 1$  to 16,  $p_\alpha$  is the conjugate momentum to  $\theta^\alpha$ ,  $\lambda^\alpha$  and  $\bar{\lambda}_\alpha$  are bosonic Weyl and anti-Weyl pure spinors constrained to satisfy  $\lambda \gamma^m \lambda = 0$  and  $\bar{\lambda} \gamma^m \bar{\lambda} = 0$ , and  $r_\alpha$  is a fermionic spinor constrained to satisfy  $\bar{\lambda} \gamma^m r = 0$ . Because of the constraints on the pure spinor variables, their conjugate momenta  $w_\alpha$ ,  $\bar{w}^\alpha$  and  $s^\alpha$  can only appear in gauge-invariant combinations such as

$$N^{mn} = \frac{1}{2} (w \gamma^{mn} \lambda), \quad J_\lambda = (w \lambda), \quad S^{mn} = \frac{1}{2} (s \gamma^{mn} \bar{\lambda}), \quad S = (s \bar{\lambda}), \quad (2.2)$$

which commute with the pure spinor constraints.

The d=10 superspace variables satisfy the free-field OPE's

$$x^m(y) x^n(z) \rightarrow -\eta^{mn} \log |y - z|^2, \quad p_\alpha(y) \theta^\beta(z) \rightarrow (y - z)^{-1} \delta_\alpha^\beta, \quad (2.3)$$

and, as long as the pure spinor conjugate momenta appear in gauge-invariant combinations and normal-ordering contributions are ignored, one can use the free-field OPE's of pure spinor variables

$$w_\alpha(y) \lambda^\beta(z) \rightarrow (y - z)^{-1} \delta_\alpha^\beta, \quad \bar{w}^\alpha(y) \bar{\lambda}_\beta(z) \rightarrow (y - z)^{-1} \delta_\beta^\alpha, \quad s^\alpha(y) r_\beta(z) \rightarrow (y - z)^{-1} \delta_\beta^\alpha. \quad (2.4)$$

It is convenient to define the spacetime supersymmetric combinations

$$\Pi^m = \partial x^m + \frac{1}{2}(\theta\gamma^m\partial\theta), \quad d_\alpha = p_\alpha - \frac{1}{2}(\partial x^m + \frac{1}{4}(\theta\gamma^m\partial\theta))(\gamma_m\theta)_\alpha \quad (2.5)$$

which satisfy the OPE's

$$d_\alpha(y)d_\beta(z) \rightarrow -(y-z)^{-1}\Pi_m\gamma_{\alpha\beta}^m, \quad d_\alpha(y)\Pi^m(z) \rightarrow (y-z)^{-1}(\gamma^m\partial\theta)_\alpha. \quad (2.6)$$

As shown in [2], the non-minimal BRST current forms a twisted  $\hat{c} = 3$  N=2 superconformal algebra with the stress tensor, a composite  $b$  ghost, and a U(1) ghost-number current. These twisted N=2 generators are

$$T = -\frac{1}{2}\partial x^m\partial x_m - p_\alpha\partial\theta^\alpha + w_\alpha\partial\lambda^\alpha + \bar{w}^\alpha\partial\bar{\lambda}_\alpha - s^\alpha\partial r_\alpha, \quad (2.7)$$

$$b = s^\alpha\partial\bar{\lambda}_\alpha + \frac{\bar{\lambda}_\alpha(2\Pi^m(\gamma_m d)^\alpha - N_{mn}(\gamma^{mn}\partial\theta)^\alpha - J_\lambda\partial\theta^\alpha - \frac{1}{4}\partial^2\theta^\alpha)}{4(\bar{\lambda}\lambda)} \quad (2.8)$$

$$- \frac{(\bar{\lambda}\gamma^{mnp}r)(d\gamma_{mnp}d + 24N_{mn}\Pi_p)}{192(\bar{\lambda}\lambda)^2} + \frac{(r\gamma_{mnp}r)(\bar{\lambda}\gamma^m d)N^{np}}{16(\bar{\lambda}\lambda)^3} - \frac{(r\gamma_{mnp}r)(\bar{\lambda}\gamma^{pqr}r)N^{mn}N_{qr}}{128(\bar{\lambda}\lambda)^4},$$

$$j_{BRST} = \lambda^\alpha d_\alpha - \bar{w}^\alpha r_\alpha, \quad (2.9)$$

$$J_{ghost} = w_\alpha\lambda^\alpha - s^\alpha r_\alpha - 2(\lambda\bar{\lambda})^{-1}[(\lambda\partial\bar{\lambda}) + (r\partial\theta)] + 2(\lambda\bar{\lambda})^{-2}(\lambda r)(\bar{\lambda}\partial\theta). \quad (2.10)$$

The terms  $-\frac{1}{16}(\lambda\bar{\lambda})^{-1}\partial^2\theta^\alpha$  in (2.8) and  $-2(\lambda\bar{\lambda})^{-1}[(\lambda\partial\bar{\lambda}) + (r\partial\theta)] + 2(\lambda\bar{\lambda})^{-2}(\lambda r)(\bar{\lambda}\partial\theta)$  in (2.10) are higher-order in  $\alpha'$  and come from normal-ordering contributions. To simplify the analysis, these normal-ordering contributions will be ignored throughout this paper. However, it should be possible to do a more careful analysis which takes into account these contributions.

### 3. Simplification of $b$ Ghost

In this section, the complicated expression of (2.8) for the  $b$  ghost will be simplified by including an auxiliary fermionic vector variable which will be later related to the RNS  $\psi^m$  variable. The trick to simplifying the  $b$  ghost is to observe that the terms involving  $d_\alpha$  in (2.8) always appear in the combination

$$\bar{\Gamma}^m = \frac{1}{2}(\lambda\bar{\lambda})^{-1}(\bar{\lambda}\gamma^m d) - \frac{1}{8}(\lambda\bar{\lambda})^{-2}(\bar{\lambda}\gamma^{mnp}r)N_{np}. \quad (3.1)$$

Note that only five components of  $\bar{\Gamma}^m$  are independent since  $\bar{\Gamma}^m(\gamma_m \bar{\lambda})^\alpha = 0$ . In terms of  $\bar{\Gamma}^m$ ,

$$b = \Pi^m \bar{\Gamma}_m - \frac{1}{4}(\lambda \bar{\lambda})^{-1}(\lambda \gamma^{mn} r) \bar{\Gamma}_m \bar{\Gamma}_n + s^\alpha \partial \bar{\lambda}_\alpha + w_\alpha \partial \theta^\alpha - \frac{1}{2}(\lambda \bar{\lambda})^{-1}(w \gamma_m \bar{\lambda})(\lambda \gamma^m \partial \theta) \quad (3.2)$$

where terms coming from normal-ordering are being ignored and the identity

$$\delta_\beta^\gamma \delta_\alpha^\delta = \frac{1}{2} \gamma_{\alpha\beta}^m \gamma_m^{\gamma\delta} - \frac{1}{8} (\gamma^{mn})_\alpha^\gamma (\gamma_{mn})_\beta^\delta - \frac{1}{4} \delta_\alpha^\gamma \delta_\beta^\delta \quad (3.3)$$

has been used.

It is useful to treat (3.1) as a first-class constraint where  $\bar{\Gamma}^m$  is a new worldsheet variable which carries +1 conformal weight and satisfies the constraint  $\bar{\Gamma}^m(\gamma_m \bar{\lambda})^\alpha = 0$ . Its conjugate momentum will be defined as  $\Gamma_m$  of conformal weight zero and can only appear in combinations invariant under the gauge transformation generated by the constraint of (3.1). Note that  $\bar{\Gamma}^m$  and  $\Gamma_m$  satisfy the OPE  $\bar{\Gamma}^m(y) \Gamma^n(z) \rightarrow (y-z)^{-1} \eta^{mn}$  and have no singular OPE's with the other variables.

One can easily verify that the  $b$  ghost of (3.2) is gauge-invariant since it has no singularity with (3.1). Furthermore, any operator  $\mathcal{O}$  which is independent of  $\Gamma_m$  can be written in a gauge-invariant manner by defining  $\mathcal{O}_{inv} = e^R \mathcal{O} e^{-R}$  where

$$R = \int \Gamma_m \left[ \frac{1}{2} (\lambda \bar{\lambda})^{-1} (\bar{\lambda} \gamma^m d) - \frac{1}{8} (\lambda \bar{\lambda})^{-2} (\bar{\lambda} \gamma^{mnp} r) N_{np} \right]. \quad (3.4)$$

For example, the gauge-invariant version of the BRST current is

$$\begin{aligned} G^+ &= e^R (\lambda^\alpha d_\alpha - \bar{w}^\alpha r_\alpha) e^{-R} = \lambda^\alpha d_\alpha - \bar{w}^\alpha r_\alpha \\ &\quad - \frac{1}{2} \Gamma^m (\lambda \bar{\lambda})^{-1} [(\bar{\lambda} \gamma_m \gamma_n \lambda) \Pi^n - (r \gamma_n \gamma_m \lambda) \bar{\Gamma}^n] \\ &\quad + \frac{1}{4} \Gamma^m \Gamma^n [(\lambda \bar{\lambda})^{-1} (\bar{\lambda} \gamma_{mn} \partial \theta) - (\lambda \bar{\lambda})^{-2} (\bar{\lambda} \partial \theta) (\bar{\lambda} \gamma_{mn} \lambda)] \\ &\quad + \frac{1}{8} \Gamma^m \Gamma^n (\lambda \bar{\lambda})^{-2} [(\bar{\lambda} \gamma_{mnp} r) \Pi^p + (r \gamma_{mnp} r) \bar{\Gamma}^p] \\ &\quad - \frac{1}{24} \Gamma^m \Gamma^n \Gamma^p [2(\lambda \bar{\lambda})^{-3} (\bar{\lambda} \partial \theta) (\bar{\lambda} \gamma_{mnp} r) - (\lambda \bar{\lambda})^{-2} (\bar{\lambda} \gamma_{mnp} \partial \bar{\lambda})] \end{aligned} \quad (3.5)$$

where the constraint of (3.1) has been used to substitute  $\bar{\Gamma}^m$  for  $\frac{1}{2}(\lambda \bar{\lambda})^{-1}(\bar{\lambda} \gamma^m d) - \frac{1}{8}(\lambda \bar{\lambda})^{-2}(\bar{\lambda} \gamma^{mnp} r) N_{np}$ .

One can also compute the gauge-invariant version of the stress tensor and U(1) current of (2.7) and (2.10) which are

$$\begin{aligned} T &= e^R \left( -\frac{1}{2} \partial x^m \partial x_m - p_\alpha \partial \theta^\alpha + w_\alpha \partial \lambda^\alpha - s^\alpha \partial r_\alpha + \bar{w}^\alpha \partial \bar{\lambda}_\alpha \right) e^{-R} \\ &= -\frac{1}{2} \partial x^m \partial x_m - p_\alpha \partial \theta^\alpha + w_\alpha \partial \lambda^\alpha - s^\alpha \partial r_\alpha + \bar{w}^\alpha \partial \bar{\lambda}_\alpha - \bar{\Gamma}^m \partial \Gamma_m \end{aligned} \quad (3.6)$$

and

$$J = e^R (w_\alpha \lambda^\alpha + r_\alpha s^\alpha) e^{-R} = w_\alpha \lambda^\alpha + r_\alpha s^\alpha + \Gamma_m \bar{\Gamma}^m. \quad (3.7)$$

The operators of (3.6), (3.2), (3.5) and (3.7) form a set of twisted N=2 superconformal generators which preserve the first-class constraint of (3.1). The resulting N=2 superconformal field theory will be related to a dynamical twisting of the RNS formalism where the RNS fermionic vector variable  $\psi^m$  is defined as

$$\psi^m = \bar{\Gamma}^m + \frac{1}{2} (\lambda \bar{\lambda})^{-1} \Gamma_n (\lambda \gamma^m \gamma^n \bar{\lambda}). \quad (3.8)$$

Note that  $\psi^m$  satisfies the usual OPE  $\psi^m(y) \psi^n(z) \rightarrow (y-z)^{-1} \eta^{mn}$  and commutes with the constraint  $\bar{\Gamma}^m (\gamma_m \bar{\lambda})^\alpha = 0$ . Since this constraint eliminates half of the  $\bar{\Gamma}^m$  variables and can be used to gauge-fix half of the  $\Gamma_m$  variables, the remaining 10 variables of  $\bar{\Gamma}^m$  and  $\Gamma_m$  can be expressed in terms of  $\psi^m$ .

Although  $w_\alpha$  and  $\bar{w}^\alpha$  have singular OPE's with  $\psi^m$ , one can define variables  $w'_\alpha$  and  $\bar{w}'^\alpha$  which have no singular OPE's with  $\psi^m$  as

$$w_\alpha = w'_\alpha - \frac{1}{4} \psi_m \psi_n [(\lambda \bar{\lambda})^{-1} (\gamma^{mn} \bar{\lambda})_\alpha - \bar{\lambda}_\alpha (\lambda \bar{\lambda})^{-2} (\lambda \gamma^{mn} \bar{\lambda})], \quad (3.9)$$

$$\bar{w}^\alpha - \frac{1}{2} \bar{\Gamma}^m \Gamma^n (\lambda \bar{\lambda})^{-1} (\gamma_m \gamma_n \lambda)^\alpha = \bar{w}'^\alpha - \frac{1}{4} \psi_m \psi_n [(\lambda \bar{\lambda})^{-1} (\gamma^{mn} \lambda)^\alpha - \lambda^\alpha (\lambda \bar{\lambda})^{-2} (\bar{\lambda} \gamma^{mn} \lambda)].$$

Note that  $\bar{w}^\alpha$  always appears in the combination  $\bar{w}^\alpha - \frac{1}{2} \bar{\Gamma}^m \Gamma^n (\lambda \bar{\lambda})^{-1} (\gamma_m \gamma_n \lambda)^\alpha$  since it is this combination which commutes with the constraint  $\bar{\Gamma}^m (\gamma_m \bar{\lambda})^\alpha = 0$ .

When expressed in terms of  $\psi^m$ ,  $w'_\alpha$  and  $\bar{w}'^\alpha$ , the twisted N=2 generators of (3.6), (3.2), (3.5) and (3.7) take the form

$$\begin{aligned} T &= -\frac{1}{2} \partial x^m \partial x_m - p_\alpha \partial \theta^\alpha + w'_\alpha \partial \lambda^\alpha - s^\alpha \partial r_\alpha + \bar{w}'^\alpha \partial \bar{\lambda}_\alpha \\ &\quad - \frac{1}{2} \psi^m \partial \psi_m - \frac{1}{4} \partial [(\lambda \bar{\lambda})^{-1} (\lambda \gamma_m \gamma_n \bar{\lambda}) \psi^m \psi^n], \end{aligned} \quad (3.10)$$

$$G^- = \frac{1}{2}(\lambda\bar{\lambda})^{-1}(\lambda\gamma_m\gamma_n\bar{\lambda})\psi^m\Pi^n + s^\alpha\partial\bar{\lambda}_\alpha + w'_\alpha\partial\theta^\alpha - \frac{1}{2}(\lambda\bar{\lambda})^{-1}(w'\gamma^m\bar{\lambda})(\lambda\gamma_m\partial\theta) \\ + \frac{1}{4}\psi_m\psi_n(\lambda\bar{\lambda})^{-1}[(\bar{\lambda}\gamma^{mn}\partial\theta) + (\lambda\bar{\lambda})^{-1}(\bar{\lambda}\partial\theta)(\lambda\gamma^{mn}\bar{\lambda}) + (r\gamma^{mn}\lambda) + (\lambda\bar{\lambda})^{-1}(r\lambda)(\lambda\gamma^{mn}\bar{\lambda})],$$

$$G^+ = -\frac{1}{2}(\lambda\bar{\lambda})^{-1}(\lambda\gamma_m\gamma_n\bar{\lambda})\psi^n\Pi^m + \lambda^\alpha d_\alpha - \bar{w}'^\alpha r_\alpha \\ + \frac{1}{4}\psi_m\psi_n(\lambda\bar{\lambda})^{-1}[(\bar{\lambda}\gamma^{mn}\partial\theta) + (\lambda\bar{\lambda})^{-1}(\bar{\lambda}\partial\theta)(\lambda\gamma^{mn}\bar{\lambda}) + (r\gamma^{mn}\lambda) + (\lambda\bar{\lambda})^{-1}(r\lambda)(\lambda\gamma^{mn}\bar{\lambda})], \\ + G^-[\frac{1}{24}(\lambda\bar{\lambda})^{-2}(\bar{\lambda}\gamma_{mnp}r)\psi^m\psi^n\psi^p],$$

$$J = -\frac{1}{2}(\lambda\bar{\lambda})^{-1}(\lambda\gamma_{mn}\bar{\lambda})\psi^m\psi^n + w'_\alpha\lambda^\alpha + r_\alpha s^\alpha,$$

where  $G^-[\frac{1}{24}(\lambda\bar{\lambda})^{-2}(\bar{\lambda}\gamma_{mnp}r)\psi^m\psi^n\psi^p]$  denotes the single pole in the OPE of  $G^-$  with  $\frac{1}{24}(\lambda\bar{\lambda})^{-2}(\bar{\lambda}\gamma_{mnp}r)\psi^m\psi^n\psi^p$  and is equal to the last two lines of (3.5).

Except for the extra term  $G^-[\frac{1}{24}(\lambda\bar{\lambda})^{-2}(\bar{\lambda}\gamma_{mnp}r)\psi^m\psi^n\psi^p]$  in  $G^+$ , the generators of (3.10) have a very symmetric form. This asymmetry in  $G^+$  and  $G^-$  can be removed by performing the similarity transformation  $\mathcal{O} \rightarrow e^R \mathcal{O} e^{-R}$  on all operators where

$$R = -\frac{1}{24} \int (\lambda\bar{\lambda})^{-2}(\bar{\lambda}\gamma_{mnp}r)\psi^m\psi^n\psi^p. \quad (3.11)$$

This similarity transformation leaves  $G^+$  of (3.10) invariant but transforms  $T$ ,  $G^-$  and  $J$  as

$$T \rightarrow T + \frac{1}{24}\partial((\lambda\bar{\lambda})^{-2}(\bar{\lambda}\gamma_{mnp}r)\psi^m\psi^n\psi^p), \quad (3.12) \\ G^- \rightarrow G^- + G^-[\frac{1}{24}(\lambda\bar{\lambda})^{-2}(\bar{\lambda}\gamma_{mnp}r)\psi^m\psi^n\psi^p], \\ J \rightarrow J + \frac{1}{12}(\lambda\bar{\lambda})^{-2}(\bar{\lambda}\gamma_{mnp}r)\psi^m\psi^n\psi^p.$$

It also transforms the constraint of (3.1) into the constraint

$$\frac{1}{2}(\lambda\bar{\lambda})^{-1}(\lambda\gamma^n\gamma^m\bar{\lambda})\psi_n = \frac{1}{2}(\lambda\bar{\lambda})^{-1}(\bar{\lambda}\gamma^m d) - \frac{1}{8}(\lambda\bar{\lambda})^{-2}(\bar{\lambda}\gamma^{mnp}r)N'_{np} \quad (3.13)$$

where  $N'_{np} = \frac{1}{2}w'\gamma_{np}\lambda$ .

After performing the similarity transformation of (3.11), the twisted N=2 generators preserve the constraint of (3.13) and take the symmetrical form

$$T = -\frac{1}{2}\partial x^m \partial x_m - \frac{1}{2}\psi^m \partial \psi_m - p_\alpha \partial \theta^\alpha + \frac{1}{2}(w'_\alpha \partial \lambda^\alpha - \lambda^\alpha \partial w'_\alpha) \quad (3.14)$$

$$-\frac{1}{2}(s^\alpha \partial r_\alpha + r_\alpha \partial s^\alpha) + \bar{w}'^\alpha \partial \bar{\lambda}_\alpha + \frac{1}{2}\partial J,$$

$$-G^+ + G^- = \psi_m \Pi^m - \lambda^\alpha d_\alpha + \bar{w}'^\alpha r_\alpha + s^\alpha \partial \bar{\lambda}_\alpha + w'_\alpha \partial \theta^\alpha - \frac{1}{2}(\lambda \bar{\lambda})^{-1}(w' \gamma^m \bar{\lambda})(\lambda \gamma_m \partial \theta),$$

$$J = -\frac{1}{2}(\lambda \bar{\lambda})^{-1}(\lambda \gamma_{mn} \bar{\lambda})\psi^m \psi^n + \frac{1}{12}(\lambda \bar{\lambda})^{-2}(\bar{\lambda} \gamma_{mnp} r)\psi^m \psi^n \psi^p + w'_\alpha \lambda^\alpha + r_\alpha s^\alpha,$$

$$G^+ + G^- = [-G^+ + G^-, J]$$

$$= \psi_m \Pi_n (\lambda \bar{\lambda})^{-1}(\lambda \gamma^{mn} \bar{\lambda}) + \lambda^\alpha d_\alpha - \bar{w}'^\alpha r_\alpha + s^\alpha \partial \bar{\lambda}_\alpha + w'_\alpha \partial \theta^\alpha - \frac{1}{2}(\lambda \bar{\lambda})^{-1}(w' \gamma^m \bar{\lambda})$$

$$+ \frac{1}{2}\psi_m \psi_n (\lambda \bar{\lambda})^{-1}[(\bar{\lambda} \gamma^{mn} \partial \theta) + (\lambda \bar{\lambda})^{-1}(\bar{\lambda} \partial \theta)(\lambda \gamma^{mn} \bar{\lambda}) + (r \gamma^{mn} \lambda) + (\lambda \bar{\lambda})^{-1}(r \lambda)](\lambda \gamma^{mn} \bar{\lambda})]$$

$$+ \frac{1}{4}\psi^m \psi^n [(\lambda \bar{\lambda})^{-2}(\bar{\lambda} \gamma_{mnp} r)\Pi^p + \frac{1}{2}(\lambda \bar{\lambda})^{-3}(r \gamma_{mnp} r)(\bar{\lambda} \gamma^p \gamma^q \lambda)\psi_q]$$

$$+ \frac{1}{12}\psi^m \psi^n \psi^p [-2(\lambda \bar{\lambda})^{-3}(\bar{\lambda} \partial \theta)(\bar{\lambda} \gamma_{mnp} r) + (\lambda \bar{\lambda})^{-2}(\bar{\lambda} \gamma_{mnp} \partial \bar{\lambda})],$$

where the last two lines in  $G^+ + G^-$  is  $G^- [\frac{1}{12}(\lambda \bar{\lambda})^{-2}(\bar{\lambda} \gamma_{mnp} r)\psi^m \psi^n \psi^p]$ . These N=2 generators of (3.14) will now be related to a dynamically twisted version of the RNS formalism.

#### 4. Dynamical Twisting of the RNS Formalism

In this section, the RNS formalism will be “dynamically twisted” to an N=2 superconformal field theory by introducing bosonic pure spinor variables  $\lambda^\alpha$  and  $\bar{\lambda}_\alpha$  and their fermionic worldsheet superpartners. The corresponding twisted N=2 superconformal generators will then be related to the twisted N=2 generators of (3.14) in the pure spinor formalism.

Twisting the N=1 RNS superconformal generators

$$T = -\frac{1}{2}\partial x^m \partial x_m - \frac{1}{2}\psi^m \partial \psi_m, \quad G = \psi^m \partial x_m \quad (4.1)$$



into N=2 superconformal generators usually involves choosing a U(5) subgroup of the Wick-rotated  $SO(10)$  Lorentz group and splitting the ten  $x^m$  and  $\psi^m$  variables into five complex pairs  $(x^a, \bar{x}^{\bar{a}})$  and  $(\psi^a, \bar{\psi}^{\bar{a}})$  for  $a = 1$  to 5. One then defines the twisted N=2 superconformal generators as

$$T_{RNS} = -\partial x^a \partial \bar{x}^{\bar{a}} - \bar{\psi}^{\bar{a}} \partial \psi^a, \quad (4.2)$$

$$G_{RNS}^- = \bar{\psi}^{\bar{a}} \partial x^a, \quad G_{RNS}^+ = -\psi^a \partial \bar{x}^{\bar{a}},$$

$$J_{RNS} = -\bar{\psi}^{\bar{a}} \psi^a,$$

which satisfy the OPE  $G^+(y)G^-(z) \rightarrow (y-z)^{-2}J(z) + (y-z)^{-1}T(z)$ .

To dynamically twist, one instead introduces pure spinor worldsheet variables  $\lambda^\alpha$  and  $\bar{\lambda}_\alpha$  satisfying

$$\lambda \gamma^m \lambda = 0, \quad \bar{\lambda} \gamma^m \bar{\lambda} = 0, \quad (4.3)$$

whose projective components parameterize the coset  $SO(10)/U(5)$ . The N=2 superconformal generators of (4.2) can then be written in a Lorentz-covariant manner as

$$T_{RNS} = -\frac{1}{2} \partial x^m \partial x_m - \frac{1}{2} \psi^m \partial \psi_m - \frac{1}{4} \partial [(\lambda \bar{\lambda})^{-1} (\lambda \gamma^m \gamma^n \bar{\lambda}) \psi_m \psi_n], \quad (4.4)$$

$$G_{RNS}^- = \frac{1}{2} (\lambda \bar{\lambda})^{-1} (\lambda \gamma^m \gamma^n \bar{\lambda}) \psi_m \partial x_n, \quad G_{RNS}^+ = -\frac{1}{2} (\lambda \bar{\lambda})^{-1} (\lambda \gamma^n \gamma^m \bar{\lambda}) \psi_m \partial x_n,$$

$$J_{RNS} = -\frac{1}{2} (\lambda \bar{\lambda})^{-1} (\lambda \gamma^m \gamma^n \bar{\lambda}) \psi_m \psi_n.$$

The next step is to introduce the fermionic worldsheet superpartners of the pure spinor variables  $(\lambda^\alpha, \bar{\lambda}_\alpha)$  and their conjugate momenta  $(w'_\alpha, \bar{w}'^\alpha)$ . The fermionic superpartners of  $\lambda^\alpha$  and  $w'_\alpha$  will be denoted  $\tilde{\theta}^\alpha$  and  $\tilde{p}_\alpha$ , and the fermionic superpartners of  $\bar{\lambda}_\alpha$  and  $\bar{w}'^\alpha$  will be denoted  $r_\alpha$  and  $s^\alpha$ . They are constrained to satisfy

$$\lambda \gamma^m \partial \tilde{\theta} = 0, \quad \bar{\lambda} \gamma^m r = 0, \quad (4.5)$$

which will be the worldsheet supersymmetry transformation of the pure spinor constraints of (4.3). Because of the constraint  $\lambda \gamma^m \partial \tilde{\theta} = 0$ ,  $\tilde{\theta}^\alpha$  is a constrained version of  $\theta^\alpha$  which only contains eleven independent non-zero modes. The corresponding twisted N=2 superconformal generators for these pure spinor multiplets are defined as

$$T_{pure} = w'_\alpha \partial \lambda^\alpha - \tilde{p}_\alpha \partial \tilde{\theta}^\alpha + \bar{w}'^\alpha \partial \bar{\lambda}_\alpha - s^\alpha \partial r_\alpha, \quad (4.6)$$

$$G_{pure}^- = w'_\alpha \partial \tilde{\theta}^\alpha + s^\alpha \partial \bar{\lambda}_\alpha, \quad G_{pure}^+ = \lambda^\alpha \tilde{p}_\alpha - \bar{w}'_\alpha r^\alpha,$$

$$J_{pure} = w'_\alpha \lambda^\alpha + r^\alpha s_\alpha,$$

which preserve the pure spinor constraints of (4.3) and (4.5).

Finally, one adds the N=2 superconformal generators of (4.4) and (4.6) in a manner that preserves the N=2 algebra. This can be done by defining  $T$ ,  $J$  and  $-G^+ + G^-$  as the sum

$$\begin{aligned} T &= T_{RNS} + T_{pure}, \quad J = J_{RNS} + J_{pure}, \\ -G^+ + G^- &= (-G^+ + G^-)_{RNS} + (-G^+ + G^-)_{pure}, \end{aligned} \quad (4.7)$$

and then defining  $G^+ + G^-$  using the commutator algebra

$$G^+ + G^- = [-G^+ + G^-, J].$$

Since  $G_{pure}^+$  and  $G_{pure}^-$  do not commute with  $J_{RNS}$ ,  $G^+ + G^-$  is not the sum of  $(G^+ + G^-)_{RNS}$  and  $(G^+ + G^-)_{pure}$ .

The resulting N=2 superconformal generators for the dynamically twisted RNS formalism are

$$T = -\frac{1}{2} \partial x^m \partial x_m - \frac{1}{2} \psi^m \partial \psi_m - \tilde{p}_\alpha \partial \tilde{\theta}^\alpha + \frac{1}{2} (w'_\alpha \partial \lambda^\alpha - \lambda^\alpha \partial w'_\alpha) \quad (4.8)$$

$$-\frac{1}{2} (s^\alpha \partial r_\alpha + r_\alpha \partial s^\alpha) + \bar{w}'^\alpha \partial \bar{\lambda}_\alpha + \frac{1}{2} \partial J,$$

$$-G^+ + G^- = \psi^m \partial x_m - \lambda^\alpha \tilde{p}_\alpha + \bar{w}'^\alpha r_\alpha + s^\alpha \partial \bar{\lambda}_\alpha + w'_\alpha \partial \tilde{\theta}^\alpha,$$

$$J = -\frac{1}{2} (\lambda \bar{\lambda})^{-1} (\lambda \gamma_{mn} \bar{\lambda}) \psi^m \psi^n + w'_\alpha \lambda^\alpha + r_\alpha s^\alpha,$$

$$G^+ + G^- = [-G^+ + G^-, J]$$

$$\begin{aligned} &= \psi_m \partial x_n (\lambda \bar{\lambda})^{-1} (\lambda \gamma^{mn} \bar{\lambda}) + \lambda^\alpha \tilde{p}_\alpha - \bar{w}'^\alpha r_\alpha + s^\alpha \partial \bar{\lambda}_\alpha + w'_\alpha \partial \tilde{\theta}^\alpha \\ &+ \frac{1}{2} \psi_m \psi_n (\lambda \bar{\lambda})^{-1} [(\bar{\lambda} \gamma^{mn} \partial \tilde{\theta}) + (\lambda \bar{\lambda})^{-1} (\bar{\lambda} \partial \tilde{\theta}) (\lambda \gamma^{mn} \bar{\lambda}) + (r \gamma^{mn} \lambda) + (\lambda \bar{\lambda})^{-1} (r \lambda) (\lambda \gamma^{mn} \bar{\lambda})]. \end{aligned}$$

The N=2 superconformal generators of (4.8) are obviously closely related to the N=2 generators of (3.14) in the pure spinor formalism, but there are three important differences. Firstly, the generators of (4.8) are not manifestly spacetime supersymmetric since they

involve  $\partial x^m$  and  $\tilde{p}_\alpha$  instead of  $\Pi^m$  and  $d_\alpha$ . Secondly, the  $U(1)$  generator  $J$  of (4.8) does not include the term  $\frac{1}{12}(\lambda\bar{\lambda})^{-2}(\bar{\lambda}\gamma_{mnp}r)\psi^m\psi^n\psi^p$ . And thirdly, the  $\tilde{\theta}^\alpha$  variable in (4.8) is constrained to satisfy  $\lambda\gamma^m\partial\tilde{\theta}=0$ .

The first difference is easily removed by performing the similarity transformation  $\mathcal{O} \rightarrow e^R \mathcal{O} e^{-R}$  on all operators in (4.8) where

$$R = \frac{1}{2} \int (\lambda\gamma^m\tilde{\theta})\psi_m. \quad (4.9)$$

This similarity transformation does not affect  $T$  or  $J$  of (4.8) but transforms  $-G^+ + G^-$  into the manifestly spacetime supersymmetric expression

$$-G^+ + G^- = \psi^m \tilde{\Pi}_m - \lambda^\alpha \tilde{d}_\alpha + \bar{w}'^\alpha r_\alpha + s^\alpha \partial \bar{\lambda}_\alpha + w'_\alpha \partial \tilde{\theta}^\alpha \quad (4.10)$$

where  $\tilde{\Pi}^m = \partial x^m + \frac{1}{2}(\tilde{\theta}\gamma^m\partial\tilde{\theta})$  and  $\tilde{d}_\alpha = \tilde{p}_\alpha - \frac{1}{2}(\partial x^m + \frac{1}{4}(\tilde{\theta}\gamma_m\partial\tilde{\theta}))(\gamma_m\tilde{\theta})_\alpha$ , and transforms the  $\psi_m\partial x_n(\lambda\bar{\lambda})^{-1}(\lambda\gamma^{mn}\bar{\lambda})$  term in  $G^+ + G^-$  into  $\psi_m\tilde{\Pi}_n(\lambda\bar{\lambda})^{-1}(\lambda\gamma^{mn}\bar{\lambda})$ .

The second difference in the generators can be removed by modifying the definition of dynamical twisting in (4.4) so that the appropriate term is added to  $J$ . The generator  $-G^+ + G^- = (-G_+ + G^-)_{RNS} + (-G^+ + G^-)_{pure}$  and the untwisted stress tensor  $T - \frac{1}{2}\partial J = (T - \frac{1}{2}\partial J)_{RNS} + (T - \frac{1}{2}\partial J)_{pure}$  of (4.8) will be left unchanged. But  $J$  will be modified so that after performing the similarity transformation of (4.9), the new  $J$  includes the term  $\frac{1}{12}(\lambda\bar{\lambda})^{-2}(\bar{\lambda}\gamma_{mnp}r)\psi^m\psi^n\psi^p$ . And to preserve the  $N=2$  algebra,  $G^+ + G^-$  will be defined as the commutator  $[-G^+ + G^-, J]$  using the new  $J$ .

Since  $e^{-R} \psi^m e^R = \psi^m - \frac{1}{2}(\lambda\gamma^m\tilde{\theta})$ , this means one should modify  $J$  in (4.8) to

$$J = -\frac{1}{2}(\lambda\bar{\lambda})^{-1}(\lambda\gamma^{mn}\bar{\lambda})\psi_m\psi_n + w'_\alpha\lambda^\alpha + r_\alpha s^\alpha \quad (4.11)$$

$$+ \frac{1}{12}(\lambda\bar{\lambda})^{-2}(\bar{\lambda}\gamma^{mnp}r)(\psi_m - \frac{1}{2}(\lambda\gamma_m\tilde{\theta}))(\psi_n - \frac{1}{2}(\lambda\gamma_n\tilde{\theta}))(\psi_p - \frac{1}{2}(\lambda\gamma_p\tilde{\theta})).$$

Although this modification of  $J$  looks unnatural, it has the important consequence of breaking the abelian shift symmetry  $\tilde{\theta}^\alpha \rightarrow \tilde{\theta}^\alpha + c^\alpha$  where  $c^\alpha$  is any constant. This shift symmetry leaves invariant the generators of (4.8), but has no corresponding symmetry in the pure spinor formalism and should not be a physical symmetry.

After modifying  $J$  in this manner and performing the similarity transformation of (4.9), the generators of (4.8) coincide with the generators of (3.14) except for the restriction that  $\lambda\gamma^m\partial\tilde{\theta}=0$ . This final difference between the generators can be removed by interpreting  $\lambda\gamma^m\partial\tilde{\theta}=0$  as a partial gauge-fixing condition for the symmetry generated by

the first-class constraint of (3.13). After relaxing the restriction  $\lambda\gamma^m\partial\tilde{\theta} = 0$  and adding the term  $-\frac{1}{2}(\lambda\bar{\lambda})^{-1}(w'\gamma^m\bar{\lambda})(\lambda\gamma^m\partial\theta)$  to  $G^-$ , the generators of (4.8) coincide with those of (3.14) and therefore preserve the constraint of (3.13).

Since the generators preserve (3.13), it is consistent to interpret (4.8) as a partially gauge-fixed version of (3.14) where the symmetry generated by (3.13) is used to gauge-fix  $\lambda\gamma^m\partial\theta = 0$ . On the other hand, the original N=2 generators of (2.7) – (2.10) of the pure spinor formalism can be interpreted as a gauge-fixed version of (3.14) where the gauge-fixing condition is  $(\lambda\gamma^m\gamma^n\bar{\lambda})\psi_n = 0$ . This is easy to see since  $(\lambda\gamma^m\gamma^n\bar{\lambda})\psi_n = 0$  implies that  $R = 0$  in the similarity transformations of (3.5), (3.6) and (3.7).

## 5. Summary

In section 2, the  $b$  ghost of the pure spinor formalism was simplified by introducing the fermionic vector variable  $\bar{\Gamma}^m$  of (3.1). After expressing  $\bar{\Gamma}^m$  in terms of the RNS variable  $\psi^m$  using (3.8), the  $b$  ghost and BRST current form a symmetric set of twisted N=2 generators (3.14) which preserve the constraint of (3.13).

In section 3, the corresponding N=2 superconformal field theory was interpreted as a dynamically twisted version of the RNS formalism in which the pure spinors  $\lambda^\alpha$  and  $\bar{\lambda}_\alpha$  parameterize the  $SO(10)/U(5)$  choices of twisting. The dynamically twisted RNS generators are obtained from (3.14) using the constraint of (3.13) to gauge-fix  $\lambda\gamma^m\partial\theta = 0$ . And the twisted N=2 generators of the original pure spinor formalism are obtained from (3.14) using the constraint of (3.13) to gauge-fix  $(\lambda\gamma^m\gamma^n\bar{\lambda})\psi_n = 0$ .

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## References

- [1] N. Berkovits, “Super Poincare covariant quantization of the superstring,” JHEP **0004**, 018 (2000). [hep-th/0001035].
- [2] N. Berkovits, “Pure spinor formalism as an N=2 topological string,” JHEP **0510**, 089 (2005). [hep-th/0509120].
- [3] O. Chandia, “The b Ghost of the Pure Spinor Formalism is Nilpotent,” Phys. Lett. B **695**, 312 (2011). [arXiv:1008.1778 [hep-th]].
- [4] R. L. Jusinkas, “Nilpotency of the b ghost in the non minimal pure spinor formalism,” [arXiv:1303.3966 [hep-th]].
- [5] N. Berkovits, “Explaining the Pure Spinor Formalism for the Superstring,” JHEP **0801**, 065 (2008). [arXiv:0712.0324 [hep-th]].